



Maximum lifespan prediction of women from Modified Weibull Distribution

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Abstract

Maximum life span (or, for humans, maximum reported age at death) is a measure of the maximum amount of time one or more members of a population have been observed to survive between birth and death. The term can also denote an estimate of the maximum amount of time that a member of a given species could survive between birth and death, provided circumstances that are optimal to that member's longevity. In the absence of age-specific mortality data, the estimation of age dependent parameter $r(t)$ in this article leads us to infer that the maximum life span of women can be estimated using the scale parameter α .

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1 Introduction

It is commonly thought that the mean human lifespan is about 125 years [4], although the oldest ages of death and life expectancy are rising today [7]. By formulating age-related trends at very old ages, researchers have long looked for a better estimation tool for the overall lifespan value [2]. Many mathematical models have been tested, such as Gompertz and Weibull models [3, 8]. However, no statistical models, including the Gompertz model, have been proposed to date that can estimate the mortality rate growth over the total life span perfectly [5]. Weon et al. defined a strictly descriptive mathematical model that allows a reasonably statistical way to obtain a clear estimate of the maximum human lifespan. In this model, an extended Weibull model is proposed by exchanging the mathematical nature of the extended exponent as a function of age. [9].

The age-dependent extended exponent $\beta(t)$ has been calculated by Weon et al using the mathematical expression $\beta(t) = \beta_0 + \beta_1(t) + \beta_2(t_2) + \dots$, where the corresponding coefficients in the plot of $\beta(t)$ versus age are calculated by a regression analysis.[9]. In reliability literature, most generalized Weibull distributions were proposed to provide a better fit of some data sets than the standard Weibull two-or-three parameter model. Chen (2000) revisited a two-parameter distribution [1]. Xie et.al (2002) [10] implemented a three-parameter Weibull distribution, the so-called modified Weibull distribution, with the probability density function specified by the probability density function, which can have a bathtub-shaped or increasing failure rate function that allows it to suit real lifetime data sets. The probability density function is given by

$$f(t; \lambda, \tau, \alpha) = \lambda T \left(\frac{t}{\alpha} \right)^{\tau-1} e^{\left\{ \left(\frac{t}{\alpha} \right)^{\tau} + \lambda \alpha \left[1 - e^{\left(\frac{t}{\alpha} \right)^{\tau} } \right] \right\}} t \geq 0,$$

where the scale parameters are $\lambda > 0$ and $\alpha > 0, \tau > 0$ is the shape parameter. The corresponding functions of survival and failure rate are given by

$$S(t; \lambda, \tau, \alpha) = e^{\left(\lambda \alpha \left[1 - e^{\left(\frac{t}{\alpha} \right)^{\tau} } \right] \right)} \rightarrow (1)$$

and $h(t; \lambda, \tau, \alpha)$

$$= \mu(t) = \lambda T \left(\frac{t}{\alpha} \right)^{\tau-1} e^{\left(\frac{t}{\alpha} \right)^{\tau}} \rightarrow (2)$$

The expanded Weibull distribution's failure rate function has a bathtub form of When $\tau < 1$ and an increasing function of When $\tau \geq 1$ [10]. This distribution is mainly related to the [1] model studied by Chen (2000) with the α additional scale parameter.

The updated Weibull model is defined by this study as a predictor of women's overall life span, which assumes that the age-dependent shape parameter would be a significant feature in women's survival curves. In this paper, we try to provide a statistical rationale, using the scale parameter α , to estimate the overall life span of women from the modified Weibull model.

2. Estimation of age-dependent shape parameter in the neighbourhood of α

If the shape parameter $r(t)$ is age - dependent, then the survival function (1) takes the form

$$S(t) = e^{\left(\lambda \alpha \left[1 - e^{\left(\frac{t}{\alpha} \right)^{r(t)}} \right] \right)} \rightarrow (3)$$

Taking logarithm on both sides of (3) we get

$$\ln S(t) = \lambda \alpha \left[1 - e^{\left(\frac{t}{\alpha} \right)^{r(t)}} \right].$$

Further simplification gives

$$e^{\left(\frac{t}{\alpha} \right)^{r(t)}} = 1 - \frac{\ln S(t)}{\lambda \alpha}$$

Taking logarithm again on both sides and simplifying further, the above equation results in

$$r(t) = \frac{\ln \left(\ln \left[1 - \frac{\ln S(t)}{\lambda \alpha} \right] \right)}{\ln \left(\frac{t}{\alpha} \right)} \rightarrow (4)$$

Note that the extended Weibull distribution, when $\alpha \rightarrow \infty$, has the Weibull distribution as a special and asymptotic case, as discussed in [10]. The mathematical relationship with the survival function is defined by the mortality function $\mu(t)$ as

$$\mu(t) = -\frac{1}{S(t)} \frac{-d \ln S(t)}{dt} \rightarrow (5)$$

On account of (3), we get

$$\begin{aligned} \mu(t) &= \frac{-d}{dt} \left[\lambda \alpha \left[1 - e^{\left(\frac{t}{\alpha} \right)^{r(t)}} \right] \right] \\ &= \lambda \alpha \frac{-d}{dt} \left(e^{\left(\frac{t}{\alpha} \right)^{r(t)}} \right) \end{aligned}$$

Therefore the mortality function for the distribution is $\mu(t)$

$$= \lambda \alpha e^{\left(\frac{t}{\alpha} \right)^{r(t)}} \left(\frac{t}{\alpha} \right)^{r(t)} \left[\frac{r(t)}{t} + \ln \left(\frac{t}{\alpha} \right) \frac{dr(t)}{dt} \right] \rightarrow (6)$$

The initial definition was obtained as follows: traditional curves of human survival show (i) a rapid decrease in survival in the first few years of life, followed by (ii) a relatively steady decrease, followed by a sudden decrease in survival near death. Interestingly, the previous behavior parallels the survival role of Weibull with $r < 1$ and the above behavior tends to suit the $r \gg 1$ case. We expand $r(t)$ in Taylor series in the neighbourhood of $t = \alpha$ upto polynomial of degree two for which we need the following results proved in our previous work.

(i). $r(t)$ is finite as $t \rightarrow \alpha$.

$$(ii). \lim_{t \rightarrow \alpha} \ln \left(\frac{t}{\alpha} \right) \frac{dr(t)}{dt} = 0 \rightarrow (7)$$

$$(iii). \mu_0(\alpha) = \lambda^2 e^2 r^2(\alpha). \rightarrow (8)$$

$$(iv). \lim_{t \rightarrow \alpha} \ln \left(\frac{t}{\alpha} \right) r''(t) = \frac{\lambda^2 e^2 r^2(\alpha)}{\alpha} - \frac{2 r^2(\alpha)}{\alpha} - \frac{2 r'(\alpha)}{\alpha} + \frac{r(\alpha)}{\alpha^2} \rightarrow (9)$$

$$(v). r'(\alpha) = \frac{\lambda^2 e^2 r^2(\alpha)}{2} - \frac{r^2(\alpha)}{\alpha} + \frac{r(\alpha)}{2\alpha} \rightarrow (10)$$

$$(vi).\mu''(\alpha) < 2\mu^3(\alpha). \rightarrow (11)$$

$$(vii).\lim_{t \rightarrow \alpha} \ln\left(\frac{t}{\alpha}\right) r'''(t) < \frac{2\lambda^2 e^3 r^3(\alpha)}{\alpha} - \frac{2r^3(\alpha)}{\alpha^3} - \frac{3r''(\alpha)}{\alpha} + \frac{2r(\alpha)}{\alpha^3} \rightarrow (12)$$

$$(viii).r''(\alpha) = \frac{2\lambda^2 e^3 r^3(\alpha)}{3} - \frac{2r^3(\alpha)}{3\alpha^2} + \frac{2r(\alpha)}{3\alpha^2} - k \rightarrow (13)$$

where k is an unknown constant and $k > 0$. Using (10) and (13), the expansion of $\beta(t)$ in Taylor series in the neighbourhood of $t = \alpha$ takes the form

$$r(t) = r(\alpha) + \frac{\lambda^2 e^2 r^2(\alpha)}{2} - \frac{r^2(\alpha)}{\alpha} + \frac{r(\alpha)}{2\alpha} (t - \alpha) + \frac{\frac{2\lambda^2 e^3 r^3(\alpha)}{3} - \frac{2r^3(\alpha)}{3\alpha^2} + \frac{2r(\alpha)}{3\alpha^2} - k}{2!} (t - \alpha)^2 \rightarrow (14)$$

To determine k , let us make use of the following property. The likelihood of survival $S(t)$ is mathematically a monotonic age decay function ($\frac{ds(t)}{dt} < 0$) between $S(0) = 1$ and $S(t_m) = 0$. For this purpose, it is possible to define the form parameter with age by:

$$\frac{dr(t)}{dt} = \begin{cases} < +\epsilon(t) & t < \alpha \\ > -\epsilon(t) & t > \alpha \end{cases} \dots\dots(I)$$

Where the math restriction of $r(t)$ is $\epsilon(t) = \frac{-r(t)}{t \ln(t/\alpha)}$. The above equation indicates a statistical tendency that if the change in the likelihood of survival is reduced ($\frac{ds(t)}{dt} < 0$), the slope of the stretched exponent becomes lower than the positive value of the young age group ($t < \alpha$) and higher than the negative value of the old age

group ($t > \alpha$). Per each survival curve, the overall human lifespan (t_m) could be calculated at the statistical limit of

$$\frac{dr(t)}{dt} = \frac{-r(t)}{t \ln\left(\frac{t}{\alpha}\right)} \rightarrow (15)$$

and as $t \rightarrow t_m$ we get

$$\begin{aligned} \frac{dr(t)}{dt} / t &= t_m \\ &= -\frac{r(t_m)}{t_m \ln(t_m/\alpha)} \end{aligned}$$

From (14) we have

$$\begin{aligned} \frac{dr(t)}{dt} / t &= t_m \\ &= r(\alpha) + \frac{\lambda^2 e^2 r^2(\alpha)}{2} - \frac{r^2(\alpha)}{\alpha} + \frac{r(\alpha)}{2\alpha} \\ &+ \left[\frac{2\lambda^2 e^3 r^3(\alpha)}{3} - \frac{2r^3(\alpha)}{3\alpha^2} + \frac{2r(\alpha)}{3\alpha^2} - k \right] (t_m - \alpha) \end{aligned} \rightarrow (16)$$

Equating (16) and the above equation we get

$$\begin{aligned} \frac{-r(t_m)}{t_m \ln\left(\frac{t_m}{\alpha}\right)} &= \frac{\lambda^2 e^2 r^2(\alpha)}{2} - \frac{r^2(\alpha)}{\alpha} + \frac{r(\alpha)}{2\alpha} \\ &+ \left[\frac{2\lambda^2 e^3 r^3(\alpha)}{3} - \frac{2r^3(\alpha)}{3\alpha^2} + \frac{2r(\alpha)}{3\alpha^2} - k \right] (t_m - \alpha) \end{aligned}$$

Solving for k results in

$$k = \frac{r(t_m)}{t_m(t_m - \alpha) \ln(t_n/\alpha)} + \left[\frac{2\lambda^2 e^3 r^3(\alpha)}{3} - \frac{2r^3(\alpha)}{3\alpha^2} + \frac{2r(\alpha)}{3\alpha^2} - k \right] + \frac{\lambda^2 e^2 r^2(\alpha)}{2(t_m - \alpha)} - \frac{r^2(\alpha)}{\alpha(t_m - \alpha)} + \frac{r(\alpha)}{2\alpha(t_m - \alpha)}$$

Substitution of k into (14) finally gives

$$r(t) = r(\alpha) + \left[\frac{\lambda^2 e^2 r^2(\alpha)}{2} - \frac{r^2(\alpha)}{\alpha} + \frac{r(\alpha)}{2\alpha} \right] (t - \alpha) - \left[\frac{r(t_m)}{t_m(t_m - \alpha) \ln(t_n/\alpha)} - \frac{\lambda^2 e^2 r^2(\alpha)}{2(t_m - \alpha)} - \frac{r^2(\alpha)}{\alpha(t_m - \alpha)} + \frac{r(\alpha)}{2\alpha(t_m - \alpha)} \right] \frac{(t - \alpha)^2}{2!}$$

The above equation gives the required estimation for the shape parameter $r(t)$ in the neighbourhood of the scale parameter α .

3 Estimation of maximum lifespan from modified Weibull distribution

$$r(t_m) = r(\alpha) + \left[\frac{\lambda^2 e^2 r^2(\alpha)}{2} - \frac{r^2(\alpha)}{\alpha} + \frac{r(\alpha)}{2\alpha} \right] (t_m - \alpha) - \left[\frac{r(t_m)}{t_m(t_m - \alpha) \ln(t_n/\alpha)} + \frac{\lambda^2 e^2 r^2(\alpha)}{2(t_m - \alpha)} - \frac{r^2(\alpha)}{\alpha(t_m - \alpha)} + \frac{r(\alpha)}{2\alpha(t_m - \alpha)} \right] \frac{(t_m - \alpha)^2}{2!}$$

Simplifying further we get

$$r(t_m) + \frac{r(t_m)(t_m - \alpha)}{t_m \ln(t_m/\alpha)} = r(\alpha) + \left[\frac{\lambda^2 e^2 r^2(\alpha)}{4} - \frac{r^2(\alpha)}{2} + \frac{r(\alpha)}{4\alpha} \right] (t_m - \alpha)$$

Dividing throughout by $(t_m - \alpha)$ and neglecting the higher powers of $r(\alpha)$ results in

$$\frac{r(t_m)}{(t_m - \alpha)} + \frac{r(t_m)}{2t_m \ln(t_m/\alpha)} = \frac{r(\alpha)}{(t_m - \alpha)} + \frac{r(\alpha)}{4\alpha}$$

$$r(t_m) \left[\frac{1}{(t_m - \alpha)} + \frac{1}{2t_m \ln(t_m/\alpha)} \right] = r(\alpha) \left[\frac{1}{(t_m - \alpha)} + \frac{1}{4\alpha} \right]$$

$$r(t_m) \left[\frac{2t_m \ln(t_n/\alpha) + (t_m - \alpha)}{(t_m - \alpha) (2t_m \ln(t_m/\alpha))} \right] = r(\alpha) \left[\frac{4\alpha + (t_m - \alpha)}{4\alpha(t_m - \alpha)} \right]$$

Upon simplification we get

$$\frac{r(t_m)}{r(\alpha)} = \frac{(4\alpha + (t_m - \alpha))(2t_m \ln(t_m/\alpha))}{4\alpha(t_m - \alpha) + 2t_m \ln(t_n/\alpha)} < 1$$

$$\frac{1}{[4\alpha + (t_m - \alpha)][2t_m \ln(t_m/\alpha)]} < \frac{1}{4\alpha(2t_m \ln(t_m/\alpha)) + 4\alpha(t_m - \alpha)}$$

Thus we have

$$(t_m - \alpha)2t_m \ln(t_m/\alpha) + 4\alpha(2t_m \ln(t_m/\alpha)) < 4\alpha(2t_m \ln(t_m/\alpha)) + 4\alpha(t_m - \alpha)$$

or, equivalently

$$(t_m/\alpha) \ln(t_n/\alpha) < 2 \quad \rightarrow (17)$$

The above estimation gives the maximum lifespan of women with respect to the age dependent parameter α . This parameter may be the characteristic life span stated in the extended Weibull model, average life span as in the

Gompertz model or any other age related parameters like age at menopause for women.

Using *Mathematica version 5.1*, the numerical experiments show that when

$\alpha = 66.67$ (world average),

$t_m < 156.391$

$\alpha = 87.27$ (Japan), $t_m < 204.714$

$\alpha = 86.88$ (Switzerland), $t_m < 203.799$

$\alpha = 86.05$ (Sweden), $t_m < 201.852$.

Obviously, this result exceeds the recorded maximum life span. The world average characteristic life α is the same as the critical lifespan in the Gompertz model [6].

Conclusion:

Most bio-medical gerontologists believe that the biomedical molecular engineering will eventually extend maximum lifespan and even bring about rejuvenation. Anti-aging drugs are a potential tool for extending life. As for the future of human longevity, it is important to understand that longevity revolution had two very distinct stages – the initial stage of mortality decline at younger ages is now replaced by a new trend of preferential improvement of the oldest-old survival. This phenomenon invalidates methods of longevity forecasting based on extrapolation of long-term historical trends. The Late-life mortality deceleration law states that death rates stop to increase exponentially at advanced ages and level-off to the late-life mortality plateau. An immediate consequence from this observation is that there is no fixed upper limit to human longevity - there is no special fixed number, which separates possible and impossible values of lifespan. This conclusion is important, because it challenge the common belief in existence of a fixed maximal human life span. Women generally live longer than males-on average by six to eight years. This difference is partly due to an inherent biological advantage for the female, but it also reflects behavioral differences between men and women. Life expectancy for women also varies across regions and income levels of countries. For instance, life expectancy for women is more than 80 years in at least 35 countries. The above discussion shows that the maximum life span of

women can be predicted when the age dependent shape parameter is known. From the recorded samples it is understood that the estimation in (17) can be further improved and it will be addressed in the near future.

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